Higher Dimensional Homology Algebra III:Projective Resolutions and Derived 2-Functors in (2-SGp)

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Abstract: In this paper, we will define the derived 2-functor by projective resolution of any symmetric 2-group, and give some related properties of the derived 2-functor.

Keywords: Symmetric 2-Group; Projective Resolution; Derived 2-Functor

1 Introduction

In recent years, higher dimensional category theory has been largely developed from a series of analogies with the potential applications. For instance, in the representation theory, the representation spaces not only to be vector spaces, but also to be categories (or even higher categories)([30]), such as the representation of categorical group, algebraic group([5, 30]), using category representations to describe the topological quantum field theory([6]) and so on. In algebraic geometry, J. Lurie gives a very tractable model of $(\infty, 1)$-categories ([17, 18, 19]), and also A. Joyal's important work [20] showing that one can do category theory in quasi-categories is an essential precursor to Luries work and is unquestionably

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one of the most important recent developments in higher category theory. Lie 2-algebra gives a solution of the Zamolodchikov tetrahedron equation [7] and Lie 2-group admits self-dual solutions in five-dimensional space time in higher Yang-Mills theory and 2-form electromagnetism[8]. L.Breen's paper[4] gives an idea of how naturally 2-categorical algebra arises in the study of algebraic geometry and differential geometry, such as Lie algebroids[28], integration of Lie 2-algebra[27], which are interesting researching subjects. Higher dimensional category theory also has been applied in algebraic topology theory[18], computer science, logic etc..

In [1], A.del Río, J. Martínez-Moreno and E. M. Vitale gave the definition of cohomology categorical groups for any complex in the 2-category (2-SGp)(which is an abelian 2-category[9]) of symmetric categorical groups(we call them symmetric 2-groups) after discussing the relative kernel and relative cokernel, and constructed a long 2-exact sequence from an extension of complexes in (2-SGp). These drive us to write a series of papers to develop a homological algebra for 2-categories (2-SGp) and ($\mathcal{R}$-2-Mod)([12]).

This is the third paper of the series. In our first paper[12] of this series, we gave the definition of $\mathcal{R}$-2-module. In the second paper[13], we proved that the 2-categories (2-SGp) and ($\mathcal{R}$-2-Mod) are projective enough. In this paper, we shall give the definition of left derived 2-functor for the 2-category (2-SGp) and give a fundamental property of derived 2-functor. When we finished this paper, we found Prof. T.Pirashvili also discussed some problems about higher homological theory[25, 26].

For a symmetric 2-group, we construct a projective resolution in the 2-category (2-SGp) and prove that it is unique up to 2-chain homotopy (Proposition 2 and Theorem 1). These results are the main stones of this paper and make it possible to define left derived 2-functor in (2-SGp).

In 1-dimensional case, derived functor has many applications in many fields of mathematics, such as ring theory, algebraic topology, representation theory, algebraic geometry etc.[23, 29, 21, 14, 15, 22]. We believe that derived 2-functor should have many applications in higher dimensional category theory.

The present paper is organized as follows. In section 2, we recall some definitions in (2-SGp) such as the relative (co)kernel, relative 2-exact which are appeared

in [9, 1, 24]. By the similar method in [1], we give the definitions of homology symmetric 2-groups for a complex of symmetric 2-groups and describe them explicitly, show the induced morphisms of homology symmetric 2-groups more clearly. We also give the definition of 2-chain homotopy of two morphisms of complexes in (2-SGp) like chain homotopy in 1-dimensional case, and prove that it induces an equivalent morphisms between homology symmetric 2-groups. In section 3, we mainly give the definition of projective resolution of a symmetric 2-group and its construction(Proposition 2). In the last section, we define the left derived 2-functor and obtain our main result Theorem 2.

2 Preliminary

In this section, we review the constructions of the relative (co)kernel and the definition of relative 2-exactness of a sequence[9, 1], and then give the homology symmetric 2-groups of a complex of symmetric 2-groups similar to the cohomology 2-group given in [1].

The relative kernel[1] $(Ker(F,\varphi), e_{(F,\varphi)}, \varepsilon_{(F,\varphi)})$ of a sequence $(F,\varphi,G) : \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in (2-SGp) is a symmetric 2-group consisting of:

· An object is a pair $(A \in obj(\mathcal{A}), a : F(A) \to 0)$ such that the following diagram commutes

$$\begin{tikzcd}
G(F(A)) \arrow[rr, "G(a)"] \arrow[dr, "\varphi_A"'] & & G(0) \arrow[dl, "\simeq"] \\
& 0 &
\end{tikzcd}$$

· A morphism $f : (A,a) \to (A^{'}, a^{'})$ is a morphism $f : A \to A^{'}$ in $\mathcal{A}$ such that the following diagram commutes

$$\begin{tikzcd}
F(A) \arrow[rr, "F(f)"] \arrow[dr, "a"'] & & F(A^{'}) \arrow[dl, "a^{'}"] \\
& 0 &
\end{tikzcd}$$

· The faithful functor $e_{(F,\varphi)} : Ker(F,\varphi) \to \mathcal{A}$ is defined by $e_{(F,\varphi)}(A,a) = A$, and the natural transformation $\varepsilon_{(F,\varphi)} : F \circ e_{(F,\varphi)} \Rightarrow 0$ by $(\varepsilon_{(F,\varphi)})_{(A,a)} = a$.

The relative cokernel[1] $(Coker(\varphi, G), p_{(\varphi,G)}, \pi_{(\varphi,G)})$ of a sequence $(F, \varphi, G) : \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in (2-SGp) is a symmetric 2-group consisting of:

$\cdot$ Objects are those of $\mathcal{C}$.

$\cdot$ A morphism from X to Y is an equivalent class of pair $(B, f) : X \to Y$ with $B \in obj(\mathcal{B})$ and $f : X \to G(B) + Y$. Two morphisms $(B, f), (B', f') : X \to Y$ are equivalent if there is $A \in obj(\mathcal{A})$ and $a : B \to F(A) + B'$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f'"'] & G(B)+Y \arrow[d, "G(a)+1"] \\
G(B')+Y \arrow[d, "\simeq"'] & G(F(A)+B')+Y \arrow[d, "\simeq"] \\
0+G(B')+Y & GF(A)+G(B')+Y \arrow[l, "\varphi_A+1+1"']
\end{tikzcd}$$

$\cdot$ The essentially surjective functor $p_{(\varphi,G)} : \mathcal{C} \to Coker(\varphi, G)$ is defined by $p_{(\varphi,G)}(X) = X$, and the natural transformation $\pi_{(\varphi,G)} : p_{(\varphi,G)} \circ G \Rightarrow 0$ by $(\pi_{(\varphi,G)})_B = 1_{G(B)}$.

The universal properties of relative kernel and cokernel just like the usual ones, more details see [1].

Definition 1. ([1]) Consider the following diagram in (2-SGp)

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "L"] \arrow[rr, bend left=40, "0"] & \mathcal{A} \arrow[r, "F"] \arrow[u, phantom, "\Uparrow\alpha"] \arrow[rr, bend right=50, "0"'] & \mathcal{B} \arrow[r, "G"] \arrow[d, phantom, "\Downarrow\varphi"] \arrow[rr, bend left=40, "0"] & \mathcal{C} \arrow[r, "M"] \arrow[u, phantom, "\Uparrow\gamma"] & \mathcal{C}'
\end{tikzcd}$$

with α compatible with φ and φ compatible with γ. By the universal property of the relative kernel $Ker(G, \gamma)$, we get a factorization (F', φ') of (F, φ) through $(e_{(F,\varphi)}, \varepsilon_{(F,\varphi)})$. By the cancellation property of $e_{(F,\varphi)}$, we have a 2-morphism $\overline{\alpha}$ as in the following diagram

$$\begin{tikzcd}
\mathcal{A}' \arrow[r, "L"] \arrow[dr, "0"'] \arrow[dr, phantom, shift left=3, "\overline{\alpha}\Leftarrow"] & \mathcal{A} \arrow[r, "F"] \arrow[d, "F'"] \arrow[dr, phantom, "\varphi'\Uparrow"] & \mathcal{B} \arrow[r, "G"] \arrow[dr, phantom, "\Rightarrow\varepsilon_{(G,\gamma)}"] & \mathcal{C} \arrow[r, "M"] & \mathcal{C}' \\
 & Ker(G,\gamma) \arrow[ur, "e_{(G,\gamma)}"'] \arrow[urr, "0"'] & & &
\end{tikzcd}$$

We say that the sequence $(L, \alpha, F, \varphi, G, \gamma, M)$ is relative 2-exact in $\mathcal{B}$ if the functor F' is essentially surjective and $\overline{\alpha}$-full.

Remark 1. The equivalent definition of relative 2-exact is also given in [1].

In the following, we will omit the composition symbol $\circ$ in our diagrams.

From [10, ?], a complex of symmetric 2-groups is a diagram in (2-SGp) of the form

$$\mathcal{A}. = \cdots \xrightarrow{L_{n+1}} \mathcal{A}_n \xrightarrow{L_n} \mathcal{A}_{n-1} \xrightarrow{L_{n-1}} \mathcal{A}_{n-2} \xrightarrow{L_{n-2}} \cdots \xrightarrow{L_2} \mathcal{A}_1 \xrightarrow{L_1} \mathcal{A}_0$$

together with a family of 2-morphisms $\{\alpha_n : L_{n-1} \circ L_n \Rightarrow 0\}_{n \geq 2}$ such that, for all n, the following diagram commutes

$$\begin{tikzcd}
L_{n-1}L_nL_{n+1} \arrow[r, Rightarrow, "\alpha_n L_{n+1}"] \arrow[d, Rightarrow, "L_{n-1}\alpha_{n+1}"'] & 0L_{n+1} \arrow[d, Rightarrow, "can"] \\
L_{n-1}0 \arrow[r, Rightarrow, "can"'] & 0
\end{tikzcd}$$

We call it 2-chain complex in our papers.

Consider part of the complex

$$\begin{tikzcd}
\mathcal{A}_{n+2} \arrow[r, "L_{n+2}"] \arrow[rr, bend left, "0"] & \mathcal{A}_{n+1} \arrow[r, "L_{n+1}"] \arrow[rr, bend right, "0"'] \arrow[u, phantom, "\alpha_{n+2}\Uparrow"] & \mathcal{A}_n \arrow[r, "L_n"] \arrow[rr, bend left, "0"] \arrow[d, phantom, "\alpha_{n+1}\Downarrow"] & \mathcal{A}_{n-1} \arrow[r, "L_{n-1}"] \arrow[u, phantom, "\Uparrow\alpha_n"] & \mathcal{A}_{n-2}
\end{tikzcd}$$

Based on the properties of relative kernel $Ker(L_n, \alpha_n)$, we have the following diagram

$$\begin{tikzcd}
\mathcal{A}_{n+2} \arrow[r, "L_{n+2}"] \arrow[rr, bend left, "0"] \arrow[drr, "0"'] & \mathcal{A}_{n+1} \arrow[r, "L_{n+1}"] \arrow[dr, "L'_{n+1}"] \arrow[u, phantom, "\alpha_{n+2}\Uparrow"] \arrow[d, phantom, "\overline{\alpha_{n+2}}\Downarrow"] & \mathcal{A}_n \arrow[r, "L_n"] \arrow[rr, bend left, "0"] & \mathcal{A}_{n-1} \arrow[r, "L_{n-1}"] \arrow[u, phantom, "\Uparrow\alpha_n"] & \mathcal{A}_{n-2} \\
& & Ker(L_n, \alpha_n) \arrow[u, "e_{(L_n,\alpha_n)}"'] \arrow[ul, phantom, "\Uparrow\alpha'_{n+1}"] & &
\end{tikzcd}$$

Similar to the definition of cohomology 2-group in [1], the nth homology symmetric 2-group $\mathcal{H}_n(\mathcal{A}.)$ of the complex $\mathcal{A}.$ is defined as the relative cokernel $Coker(\overline{\alpha_{n+2}}, L'_{n+1})$.

Note that, to get $\mathcal{H}_0(\mathcal{A}.)$ and $\mathcal{H}_1(\mathcal{A}.)$, we have to complete the complex $\mathcal{A}.$ on the right with the two zero-morphisms and two canonical 2-morphisms $\cdots \xrightarrow{L_2} \mathcal{A}_1 \xrightarrow{L_1} \mathcal{A}_0 \xrightarrow{0} 0 \xrightarrow{0} 0$, can : $0 \circ L_1 \Rightarrow 0$, can : $0 \circ 0 \Rightarrow 0$.

We give an explicit description of $\mathcal{H}_n(\mathcal{A}.)$ following from the cohomology symmetric 2-group given in [1].

$\cdot$ an object of $\mathcal{H}_n(\mathcal{A}.)$ is an object of the relative kernel $Ker(L_n, \alpha_n)$, that is a pair

$$(A_n \in obj(\mathcal{A}_n), a_n : L_n(A_n) \to 0)$$

such that $L_{n-1}(a_n) = (\alpha_n)_{A_n}$;

$\cdot$ a morphism $(A_n, a_n) \to (A'_n, a'_n)$ is an equivalent pair

$$(X_{n+1} \in obj(\mathcal{A}_{n+1}), x_{n+1} : A_n \to L_{n+1}(X_{n+1}) + A'_n)$$

such that the following diagram commutes

$$\begin{tikzcd}
L_n(A_n) \arrow[r, "L_n(x_{n+1})"] \arrow[d, "a_n"'] & L_n(L_{n+1}(X_{n+1}) + A'_n) \arrow[d, "\simeq"] \\
0 & L_nL_{n+1}(X_{n+1}) + L_n(A'_n) \arrow[d, "\alpha_{n+1X_{n+1}} + 1"] \\
L_n(A'_n) \arrow[u, "a'_n"] & 0 + L_n(A'_n) \arrow[l, "\simeq"]
\end{tikzcd}$$

Two morphisms $(X_{n+1}, x_{n+1}), (X'_{n+1}, x'_{n+1}) : (A_n, a_n) \to (A'_n, a'_n)$ are equivalent if there is a pair

$$(X_{n+2} \in obj(\mathcal{A}_{n+2}), x_{n+2} : X_{n+1} \to L_{n+2}(X_{n+2}) + X'_{n+1})$$

such that the following diagram commutes

$$
\begin{tikzcd}
A_n \arrow[r, "x_{n+1}"] \arrow[ddd, "x_{n+1}'"'] & L_{n+1}(X_{n+1})+A_n' \arrow[d, "L_{n+1}(x_{n+2})+1"] \\
& L_{n+1}(L_{n+2}(X_{n+2})+X_{n+1}')+A_n' \arrow[d, "\simeq"] \\
& L_{n+1}L_{n+2}(X_{n+2})+L_{n+1}(X_{n+1}')+A_n' \arrow[d, "\alpha_{n+2X_{n+2}}+1"] \\
L_{n+1}(X_{n+1}')+A_n' & 0+L_{n+1}(X_{n+1}')+A_n' \arrow[l, "\simeq"]
\end{tikzcd}
$$

Similar to [1], a morphism $(F.,\lambda.):\mathcal{A}.\to\mathcal{B}.$ of complexes in (2-SGp) is a picture in the following diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "L_{n+2}"] & \mathcal{A}_{n+1} \arrow[r, "L_{n+1}"] \arrow[rr, bend left, "0"] \arrow[d, "F_{n+1}"'] \arrow[dr, phantom, "\lambda_{n+1}\Leftarrow"] & \mathcal{A}_n \arrow[r, "L_n"] \arrow[u, phantom, "\Uparrow\alpha_{n+1}"] \arrow[d, "F_n"'] \arrow[dr, phantom, "\lambda_n\Leftarrow"] & \mathcal{A}_{n-1} \arrow[r, "L_{n-1}"] \arrow[d, "F_{n-1}"'] \arrow[dr, phantom, "\lambda_{n-1}\Leftarrow"] & \mathcal{A}_{n-2} \arrow[r] \arrow[d, "F_{n-2}"] & \cdots \\
\cdots \arrow[r, "M_{n+2}"'] & \mathcal{B}_{n+1} \arrow[r, "M_{n+1}"'] \arrow[rr, bend right, "0"'] & \mathcal{B}_n \arrow[r, "M_n"'] \arrow[d, phantom, "\beta_{n+1}\Downarrow"] & \mathcal{B}_{n-1} \arrow[r, "M_{n-1}"'] & \mathcal{B}_{n-2} \arrow[r] & \cdots \\
& & {} & & &
\end{tikzcd}
$$

where $F_n:\mathcal{A}_n\to\mathcal{B}_n$ is 1-morphism in (2-SGp), $\lambda_n:F_{n-1}\circ L_n\Rightarrow M_n\circ F_n$ is 2-morphism in (2-SGp), for each n, making the following diagram commutative

$$
\begin{tikzcd}
F_{n-1}L_nL_{n+1} \arrow[r, Rightarrow, "\lambda_nL_n"] \arrow[d, Rightarrow, "F_{n-1}\alpha_{n+1}"'] & M_nF_nL_{n+1} \arrow[r, Rightarrow, "M_n\lambda_{n+1}"] & M_nM_{n+1}F_{n+1} \arrow[d, Rightarrow, "\beta_{n+1}F_{n+1}"] \\
F_{n-1}0 \arrow[r, Rightarrow, "can"'] & 0 & 0F_{n+1} \arrow[l, Rightarrow, "can"]
\end{tikzcd}
$$

Such a morphism induces, for each n, a morphism of homology symmetric 2-groups $\mathcal{H}_n(F.):\mathcal{H}_n(\mathcal{A}.)\to\mathcal{H}_n(\mathcal{B}.)$ from the universal properties of relative kernel and cokernel. It can be described explicitly.

Given an object $(A_n\in obj(\mathcal{A}_n), a_n:L_n(A_n)\to 0)$ of $\mathcal{H}_n(\mathcal{A}.)$ with $L_{n-1}(a_n)=(\alpha_n)_{A_n}$, we have $\mathcal{H}_n(F.)(A_n,a_n)=(F_n(A_n)\in obj(\mathcal{B}_n), b_n:M_n(F_n(A_n))\to 0)$, where b_n is the composition $M_n(F_n(A_n))\xrightarrow{(\lambda_n)^{-1}_{A_n}}F_{n-1}L_n(A_n)$ $\xrightarrow{F_{n-1}(a_n)}F_{n-1}(0)\backsimeq 0$, together with $M_{n-1}(b_n)=(\beta_n)_{F_n(A_n)}$. In fact, from the commutative diagram of λ_n, we have the following commutative diagram

$$\begin{tikzcd}
F_{n-2}L_{n-1}L_n(A_n) \arrow[r, "\lambda_{n-1L_n(A_n)}"] \arrow[d, "F_{n-2}(\alpha_{nA_n})"'] & M_{n-1}F_{n-1}L_n(A_n) \arrow[r, "M_{n-1}(\lambda_{nA_n})"] & M_{n-1}M_nF_n(A_n) \arrow[d, "\beta_{nF_n(A_n)}"] \\
F_{n-2}(0) \arrow[r, "\simeq"] & 0 & 0(F_n(A_n)) \arrow[l, "="']
\end{tikzcd}$$

Moreover, consider 2-morphism $\lambda_{n-1} : F_{n-2} \circ L_{n-1} \Rightarrow M_{n-1} \circ F_{n-1}$ and a morphism $a_n : L_n(A_n) \to 0$ in $\mathcal{A}_{n-1}$, we have the following commutative diagram

$$\begin{tikzcd}
F_{n-2}L_{n-1}L_n(A_n) \arrow[r, "\lambda_{n-1L_n(A_n)}"] \arrow[d, "F_{n-2}L_{n-1}(a_n)"'] & M_{n-1}F_{n-1}L_n(A_n) \arrow[d, "M_{n-1}(F_{n-1}(a_n))"] & \\
F_{n-2}(L_{n-1}(0)) \arrow[r, "\simeq"] & M_{n-1}(F_{n-1}(0)) \arrow[r, "\simeq"] & 0
\end{tikzcd}$$

Then, by the above two diagrams, we have $M_{n-1}(b_n) = (\beta_n)_{F_n(A_n)}$, i.e. $(F_n(A_n), b_n)$ is an object of $\mathcal{H}_n(\mathcal{B}.)$.

Given a morphism $[X_{n+1} \in obj(\mathcal{A}_{n+1}), x_{n+1} : A_n \to L_{n-1}(X_{n+1}) + A_n^{'}] : (A_n, a_n) \to (A_n^{'}, a_n^{'})$ in $\mathcal{H}_n(\mathcal{A}.)$, satisfying the condition as in the above definition. We have $\mathcal{H}_n(F.)[X_{n+1}, x_{n+1}] = [F_{n+1}(X_{n+1}) \in obj(\mathcal{B}_{n+1}), \overline{x_{n+1}} : F_n(A_n) \to M_{n+1}(F_{n+1}(X_{n+1})) + F_n(A_n^{'})] : (F_n(A_n, b_n) \to (F_n(A_n^{'}), b_n^{'})$, where $\overline{x_{n+1}}$ is the composition $F_n(A_n) \xrightarrow{F_n(x_{n+1})} F_n(L_{n+1}(X_{n+1} + A_n^{'})) \backsimeq F_nL_{n+1}(X_{n+1}) + F_n(A_n^{'}) \xrightarrow{(\lambda_{n+1})_{X_{n+1}}+1} M_{n+1}(F_{n+1}(X_{n+1})) + F_n(A_n^{'})$ and such that the following diagram commutes

$$\begin{tikzcd}
M_nF_n(A_n) \arrow[r, "M_n(\overline{x_{n+1}})"] \arrow[d, "b_n"] & M_n(M_{n+1}(F_{n+1}(X_{n+1})) + F_n(A_n^{'})) \arrow[d, "\simeq"] \\
0 & M_nM_{n+1}F_{n+1}(X_{n+1}) + M_nF_n(A_n^{'}) \arrow[d, "\beta_{n+1F_{n+1}(X_{n+1})} + 1"] \\
M_nF_n(A_n^{'}) \arrow[u, "b_n^{'}"] & 0 + M_nF_n(A_n^{'}) \arrow[l, "\simeq"]
\end{tikzcd}$$

In fact, we have the following commutative diagrams

$$\begin{tikzcd}
F_{n-1}(0) \arrow[ddd, "\simeq"] & F_{n-1}L_n(A_n) \arrow[l, "F_{n-1}(a_n)"'] \arrow[r, "\lambda_{n_{A_n}}"] \arrow[d, "F_{n-1}L_n(x_{n+1})"'] \arrow[dr, phantom, "\mathrm{I}"] & M_nF_n(A_n) \arrow[d, "M_nF_n(x_{n+1})"] \\
 & F_{n-1}L_n(L_{n+1}(X_{n+1})+A_n') \arrow[r, "\lambda_{n_{L_{n+1}(X_{n+1})+A_n'}}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\mathrm{II}"] & M_nF_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\simeq"] \\
 & F_{n-1}L_nL_{n+1}(X_{n+1})+F_{n-1}L_n(A_n') \arrow[r, "\lambda_{n_{L_{n+1}(X_{n+1})}}+\lambda_{n_{A_n'}}"'] \arrow[d, "F_{n-1}(\alpha_{n+1_{X_{n+1}}})+1"'] & M_nF_nL_{n+1}(X_{n+1})+M_nF_n(A_n') \arrow[d, "M_n(\lambda_{n+1_{X_{n+1}}})+1"] \\
0 \arrow[r, phantom, "\mathrm{V}"] & F_{n-1}(0)+F_{n-1}L_n(A_n') \arrow[d, "\simeq"'] \arrow[r, phantom, "\mathrm{III}"] & M_nM_{n+1}F_{n+1}(X_{n+1})+M_nF_n(A_n') \arrow[d, "\beta_{n+1_{F_{n+1}(X_{n+1})}}+1"] \\
 & 0+F_{n-1}L_n(A_n') \arrow[r, "1+\lambda_{n_{A_n'}}"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\mathrm{IV}"] & 0+M_nF_n(A_n') \arrow[d, "\simeq"] \\
F_{n-1}(0) \arrow[uu, "\simeq"] & F_{n-1}L_n(A_n') \arrow[l, "F_{n-1}(a_n')"] \arrow[r, "\lambda_{n_{A_n'}}"'] & M_nF_n(A_n')
\end{tikzcd}$$

The commutativity of I follows from λ_n is a natural transformation. The commutativity of II follows from λ_n is a 2-morphism. The commutativity of III follows from the commutativity of λ_n in definition. The commutativity of IV is obvious. The commutativity of V follows from the operation of F_{n-1} on the commutative diagram of $[X_{n+1}, x_{n+1}]$.

$\mathcal{H}_n(F.)$ is a morphism in (2-SGp) follows from the properties of F_n.

Remark 2. 1. For a complex of symmetric 2-groups which is relative 2-exact in each point, the (co)homology symmetric 2-groups are always zero symmetric 2-group(only one object and one morphism)([1]).

2. For morphisms $\mathcal{A}. \xrightarrow{(F.,\lambda.)} \mathcal{B}. \xrightarrow{(G.,\mu.)} \mathcal{C}.$ of complexes of symmetric 2-groups, their composite is given by $(G_n \circ F_n, (\mu_n \circ F_{n+1}) \star (G_n \circ \lambda_n))$, for $n \in \mathbb{Z}$, where $\star$ is the vertical composition of 2-morphisms in 2-category([3]). Moreover, $\mathcal{H}_n(G. \circ F.) \backsimeq \mathcal{H}_n(G.) \circ \mathcal{H}_n(F.)$ of homology symmetric 2-groups.

Definition 2. Let $(F., \lambda.), (G., \mu.) : (\mathcal{A}., L., \alpha.) \to (\mathcal{B}., M., \beta.)$ be two morphisms of 2-chain complexes of symmetric 2-groups. If there is a family of 1-morphisms $\{H_n : \mathcal{A}_n \to \mathcal{B}_{n+1}\}_{n \in \mathbb{Z}}$ and a family of 2-morphisms $\{\tau_n : F_n \Rightarrow M_{n+1} \circ H_n + H_{n-1} \circ L_n + G_n : \mathcal{A}_n \to \mathcal{B}_n\}_{n \in \mathbb{Z}}$ satisfying the obvious compatible conditions, i.e. the following diagram commutes

$$
\begin{tikzcd}
F_{n-1}L_n \arrow[rr, Rightarrow, "\lambda_n"] \arrow[d, Rightarrow, "\tau_{n-1}L_n"'] & & M_nF_n \arrow[d, Rightarrow, "M_n\tau_n"] \\
(M_nH_{n-1}+H_{n-2}L_{n-1}+G_{n-1})L_n \arrow[d, Rightarrow, "can"'] & & M_n(M_{n+1}H_n+H_{n-1}L_n+G_n) \arrow[d, Rightarrow, "can"] \\
M_nH_{n-1}L_n+H_{n-2}L_{n-1}L_n+G_{n-1}L_n \arrow[d, Rightarrow, "1+H_{n-2}\alpha_n+1"'] & & M_nM_{n+1}H_n+M_nH_{n-1}L_n+M_nG_n \arrow[d, Rightarrow, "\beta_{n+1}H_n+1+1"] \\
M_nH_{n-1}L_n+H_{n-2}(0)+G_{n-1}L_n \arrow[d, Rightarrow, "can"'] & & 0+M_nH_{n-1}L_n+M_nG_n \arrow[d, Rightarrow, "1+\mu_n^{-1}"] \\
M_nH_{n-1}L_n+0+G_{n-1}L_n \arrow[rr, Rightarrow, "can"'] & & M_nH_{n-1}L_n+G_{n-1}L_n
\end{tikzcd}
$$

We call the above morphisms $(F., \lambda.), (G., \mu.)$ are 2-chain homotopy.

Proposition 1. *Let $(F., \lambda.), (G., \mu.) : (\mathcal{A}., L., \alpha.) \to (\mathcal{B}., M., \beta.)$ be two morphisms of 2-chain complexes of symmetric 2-groups. If they are 2-chain homotopy, there is an equivalence $\mathcal{H}_n(F.) \backsimeq \mathcal{H}_n(G.)$ between induced morphisms.*

Proof. In order to prove the equivalence between two morphisms, it will suffice to construct a 2-morphism $\varphi_n : \mathcal{H}_n(F.) \Rightarrow \mathcal{H}_n(G.)$, for each n.

There are induced morphisms

$$
\begin{aligned}
\mathcal{H}_n(F.) : \mathcal{H}_n(\mathcal{A}.) &\to \mathcal{H}_n(\mathcal{B}.) \\
(A_n, a_n) &\mapsto (F_n(A_n), b_n), \\
[X_{n+1}, x_{n+1}] &\mapsto [F_{n+1}(X_{n+1}), \overline{x_{n+1}}]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}_n(G.) : \mathcal{H}_n(\mathcal{A}.) &\to \mathcal{H}_n(\mathcal{B}.) \\
(A_n, a_n) &\mapsto (G_n(A_n), \overline{b_n}), \\
[X_{n+1}, x_{n+1}] &\mapsto [G_{n+1}(X_{n+1}), \overline{x_{n+1}}']
\end{aligned}
$$

For any object (A_n, a_n) of $\mathcal{H}_n(\mathcal{A}.)$, let $Y_{n+1} = H_n(A_n)$. Consider the following composition morphism $F_n(A_n) \xrightarrow{(\tau_n)_{A_n}} (M_{n+1} \circ H_n + H_{n-1} \circ L_n + G_n)(A_n) \xrightarrow{1+H_{n-1}(a_n)+1} M_{n+1}(H_n(A_n)) + H_{n-1}(0) + G_n(A_n) \backsimeq M_{n+1}(Y_{n+1}) + 0 + G_n(A_n) \backsimeq M_{n+1}(Y_{n+1}) + G_n(A_n)$ of $\mathcal{B}_n$. We get a morphism $[Y_{n+1} \in obj(\mathcal{B}_{n+1}), y_{n+1} : F_n(A_n) \to M_{n+1}(Y_{n+1}) + G_n(A_n)] : (F_n(A_n), b_n) \to (G_n(A_n), \overline{b_n})$ of $\mathcal{H}_n(\mathcal{B}.)$ such that the following diagram commutes

$$
\begin{tikzcd}
M_n(F_n(A_n)) \arrow[r, "M_n(y_{n+1})"] \arrow[d, "b_n"'] & M_n(M_{n+1}(H_n(A_n))+G_n(A_n)) \arrow[d, "\simeq"] \\
0 & M_n(M_{n+1}(H_n(A_n)))+M_n(G_n(A_n)) \arrow[d, "\beta_{n+1 H_n(A_n)}+1"] \\
M_n(G_n(A_n)) \arrow[u, "\overline{b_n}"] & 0+M_n(G_n(A_n)) \arrow[l, "\simeq"']
\end{tikzcd}
$$

From the compatible condition of $(\tau_n)_{n\in\mathbb{Z}}$, we have the following commutative diagram

$$
\begin{tikzcd}[column sep=small]
M_n(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n) \arrow[r, "\simeq"] & M_nM_{n+1}H_n(A_n)+M_nH_{n-1}L_n(A_n)+M_nG_n(A_n) \arrow[r, "\beta_{n+1 H_n(A_n)}+1"] \arrow[d, "1+M_nH_{n-1}(a_n)+1"] & 0+M_nH_{n-1}L_n(A_n)+M_nG_n(A_n) \arrow[d, "\simeq"] \\
M_nF_n(A_n) \arrow[u, "M_n(\tau_{n A_n})"] \arrow[dr, "M_n(y_{n+1})"] \arrow[ddd, "\lambda^{-1}_{n A_n}"'] & M_nM_{n+1}H_n(A_n)+M_nH_{n-1}(0)+M_nG_n(A_n) \arrow[d, "\simeq"] & M_nH_{n-1}L_n(A_n)+M_nG_n(A_n) \arrow[d, "1+\mu^{-1}_{n A_n}"] \\
 & M_nM_{n+1}H_n(A_n)+M_nG_n(A_n) \arrow[d, "\beta_{n+1 H_n(A_n)}+1"] & M_nH_{n-1}L_n(A_n)+G_{n-1}L_n(A_n) \arrow[d, "M_nH_{n-1}(a_n)+G_{n-1}(a_n)"] \\
 & 0+M_nG_n(A_n) \arrow[d, "\mu^{-1}_{n A_n}"] & M_nH_{n-1}(0)+G_{n-1}(0) \arrow[d, "\simeq"] \\
F_{n-1}L_n(A_n) \arrow[dd, "\tau_{n-1 L_n(A_n)}"'] \arrow[ddr, "F_{n-1}(a_n)"] & G_{n-1}L_n(A_n) \arrow[d, "G_{n-1}(a_n)"] & 0+0 \arrow[ddl, "\simeq"] \\
 & G_{n-1}(0) \arrow[d, "\simeq"] & \\
(M_nH_{n-1}+H_{n-2}L_{n-1}+G_{n-1})L_n(A_n) \arrow[r, "(M_nH_{n-1}+H_{n-2}L_{n-1}+G_{n-1})(a_n)"'] \arrow[d, "\simeq"'] & 0 & \\
M_nH_{n-1}L_n(A_n)+H_{n-2}L_{n-1}L_n(A_n)+G_{n-1}L_n(A_n) \arrow[d, "1+H_{n-2}(\alpha_{n A_n})+1"'] & & \\
M_nH_{n-1}L_n(A_n)+H_{n-2}(0)+G_{n-1}L_n(A_n) \arrow[rr, "\simeq"] & & M_nH_{n-1}L_n(A_n)+0+G_{n-1}L_n(A_n) \arrow[uuuuuu, bend right, "\simeq"']
\end{tikzcd}
$$

So $[Y_{n+1}, y_{n+1}]$ is a morphism in $\mathcal{H}_n(\mathcal{B}.)$, then we can define a 2-morphism φ_n. For any morphism $[X_{n+1}, x_{n+1}] : (A_n, a_n) \to (A_n', a_n')$ in $\mathcal{H}_n(\mathcal{A}.)$, where $X_{n+1} \in obj(\mathcal{A}_{n+1})$, $x_{n+1} : A_n \to L_{n+1}(X_{n+1}) + A_n'$ satisfying the following commutative diagram

$$\begin{tikzcd}
L_n(A_n) \arrow[r, "L_n(x_{n+1})"] \arrow[d, "a_n"'] & L_n(L_{n+1}(X_{n+1})+A_n') \arrow[d] \\
0 & L_nL_{n+1}(X_{n+1})+L_n(A_n') \arrow[d, "\alpha_{nX_{n+1}}+1"] \\
L_n(A_n') \arrow[u, "a_n'"] & 0+L_n(A_n') \arrow[l]
\end{tikzcd}$$

$\mathcal{H}_n(F.)[X_{n+1}, x_{n+1}] = [F_{n+1}(X_{n+1}), \overline{x_{n+1}}]$, $\mathcal{H}_n(G.)[X_{n+1}, x_{n+1}] = [G_{n+1}(X_{n+1}), \overline{x_{n+1}}']$, where $\overline{x_{n+1}}$ and $\overline{x_{n+1}}'$ are the following composition morphisms $\overline{x_{n+1}} : F_n(A_n) \xrightarrow{F_n(x_{n+1})} F_n(L_{n+1}(X_{n+1})+A_n') \backsimeq (F_n \circ L_{n+1})(X_{n+1})+F_n(A_n') \xrightarrow{(\lambda_{n+1})_{X_{n+1}}+1} M_{n+1}(F_{n+1}(X_{n+1}))+F_n(A_n')$, $\overline{x_{n+1}}' : G_n(A_n) \xrightarrow{G_n(x_{n+1})} G_n(L_{n+1}(X_{n+1}) + A_n') \backsimeq (G_n \circ L_{n+1})(X_{n+1}) + G_n(A_n') \xrightarrow{(\mu_{n+1})_{X_{n+1}}+1} M_{n+1}(G_{n+1}(X_{n+1})) + G_n(A_n')$.

Then we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{H}_n(F.)(A_n, a_n) \arrow[r, "\varphi_{n(A_n,a_n)} = {[H_n(A_n),\, y_{n+1}]}"] \arrow[d, "{[F_{n+1}(X_{n+1}),\, \overline{x_{n+1}}]}"'] & \mathcal{H}_n(G.)(A_n, a_n) \arrow[d, "{[G_{n+1}(X_{n+1}),\, \overline{x_{n+1}}']}"] \\
\mathcal{H}_n(F.)(A_n', a_n') \arrow[r, "\varphi_{n(A_n',a_n')} = {[H_n(A_n'),\, y_{n+1}']}"] & \mathcal{H}_n(G.)(A_n', a_n')
\end{tikzcd}$$

There exist $[Y_{n+2} \triangleq H_{n+1}(X_{n+1}), y_{n+2}] : ([H_n(A_n'), y_{n+1}'] \circ [F_{n+1}(X_{n+1}), \overline{x_{n+1}}]) \to [G_{n+1}(X_{n+1}), \overline{x_{n+1}}'] \circ [H_n(A_n), y_{n+1}]$ induced by $\tau.$. In fact, $((H_n(A_n'), y_{n+1}') \circ F_{n+1}(X_{n+1}), \overline{x_{n+1}})) = [F_{n+1}(X_{n+1}) + H_n(A_n'), (1 + y_{n+1}') \circ \overline{x_{n+1}}']$, $[G_{n+1}(X_{n+1}), \overline{x_{n+1}}] \circ [H_n(A_n), y_{n+1}] = [H_n(A_n) + G_{n+1}(X_{n+1}), (1 + \overline{x_{n+1}}') \circ y_{n+1}]$ from the composition of morphisms in relative cokernel, so y_{n+2} is the composition morphism $F_{n+1}(X_{n+1}) + H_n(A_n') \xrightarrow{(\tau_{n+1})_{X_{n+1}}+1} (M_{n+2}H_{n+1} + H_nL_{n+1} + G_{n+1})(X_{n+1}) + H_n(A_n') \backsimeq M_{n+2}H_{n+1}(X_{n+1}) + H_nL_{n+1}(X_{n+1}) + G_{n+1}(X_{n+1}) + H_n(A_n') \backsimeq M_{n+2}H_{n+1}(X_{n+1}) + G_{n+1}(X_{n+1}) + H_n(L_{n+1}(X_{n+1} + A_n') \xrightarrow{1+1+H_n(x_{n+1}^{-1})} M_{n+2}H_{n+1}(X_{n+1}) + G_{n+1}(X_{n+1}) + H_n(A_n) \backsimeq M_{n+2}H_{n+1}(X_{n+1}) + H_n(A_n) + G_{n+1}(X_{n+1})$. Moreover the morphism $[Y_{n+2}, y_{n+2}]$ makes the following diagram commute

$$
\begin{tikzcd}
F_n(A_n) \arrow[r, "(1+y_{n+1}')\circ\overline{x_{n+1}}"] \arrow[d, "(1+\overline{x_{n+1}}')\circ y_{n+1}"'] & M_{n+1}(F_{n+1}(X_{n+1})+H_n(A_n'))+G_n(A_n') \arrow[d, "M_{n+1}(y_{n+2})+1"] \\
M_{n+1}(H_n(A_n)+G_{n+1}(X_{n+1}))+G_n(A_n') & M_{n+1}(M_{n+2}H_{n+1}(X_{n+1})+H_n(A_n)+G_{n+1}(X_{n+1}))+G_n(A_n') \arrow[d, "\simeq"] \\
0+M_{n+1}(H_n(A_n)+G_{n+1}(X_{n+1}))+G_n(A_n') \arrow[u, "\simeq"] & M_{n+1}M_{n+2}H_{n+1}(X_{n+1})+M_{n+1}(H_n(A_n)+G_{n+1}(X_{n+1}))+G_n(A_n') \arrow[l, bend left, "\beta_{n+2_{H_{n+1}(X_{n+1})}}+1+1"]
\end{tikzcd}
$$

for the following several commutative diagrams

$$
\begin{tikzcd}
F_n(A_n) \arrow[r, "F_n(x_{n+1})"] \arrow[d, "\tau_{n_{A_n}}"'] \arrow[dr, phantom, "\mathrm{I}"] & F_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\tau_{n_{L_{n+1}(X_{n+1})+A_n'}}"] \\
(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n) \arrow[r, "(M_{n+1}H_n+H_{n-1}L_n+G_n)(x_{n+1})"'] & (M_{n+1}H_n+H_{n-1}L_n+G_n)(L_{n+1}(X_{n+1})+A_n')
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_n(L_{n+1}(X_{n+1})+A_n') \arrow[r, "\simeq"] \arrow[d, "\tau_{n_{L_{n+1}(X_{n+1})+A_n'}}"'] \arrow[dr, phantom, "\mathrm{II}"] & F_nL_{n+1}(X_{n+1})+F_n(A_n') \arrow[d, "\tau_{n_{L_{n+1}(X_{n+1})}}+\tau_{n_{A_n'}}"] \\
(M_{n+1}H_n+H_{n-1}L_n+G_n)(L_{n+1}(X_{n+1})+A_n') \arrow[r, "\simeq"'] & (M_{n+1}H_n+H_{n-1}L_n+G_n)L_{n+1}(X_{n+1})+(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n')
\end{tikzcd}
$$

$$
\begin{tikzcd}
(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n) \arrow[r, "(M_{n+1}H_n+H_{n-1}L_n+G_n)(x_{n+1})"] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\mathrm{III}"] & (M_{n+1}H_n+H_{n-1}L_n+G_n)(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\simeq"] \\
M_{n+1}H_n(A_n)+H_{n-1}L_n(A_n)+G_n(A_n) \arrow[r, "M_{n+1}H_n(x_{n+1})+H_{n-1}L_n(x_{n+1})+G_n(x_{n+1})"'] & M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+H_{n-1}L_n(L_{n+1}(X_{n+1})+A_n')+G_n(L_{n+1}(X_{n+1})+A_n')
\end{tikzcd}
$$

$$
\begin{tikzcd}
M_{n+1}H_n(A_n)+H_{n-1}L_n(A_n)+G_n(A_n) \arrow[r, "M_{n+1}H_n(x_{n+1})+H_{n-1}L_n(x_{n+1})+G_n(x_{n+1})"] \arrow[d, "1+H_{n-1}(a_n)+1"'] & M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+H_{n-1}L_n(L_{n+1}(X_{n+1})+A_n')+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\simeq"] \\
M_{n+1}H_n(A_n)+H_{n-1}(0)+G_n(A_n) \arrow[d, "\simeq"'] & M_{n+1}H_nL_{n+1}(X_{n+1})+H_{n-1}L_nL_{n+1}(X_{n+1})+G_nL_{n+1}(X_{n+1})+(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n') \arrow[d, "1+H_{n-1}(\alpha_{n+1X_{n+1}})+1+1"] \\
M_{n+1}H_n(A_n)+G_n(A_n) \arrow[d, "1+G_n(x_{n+1})"'] \arrow[r, phantom, "\mathrm{IV}"] & M_{n+1}H_nL_{n+1}(X_{n+1})+H_{n-1}(0)+G_nL_{n+1}(X_{n+1})+(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n') \arrow[d, "\simeq"] \\
M_{n+1}H_n(A_n)+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\simeq"'] & M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+H_{n-1}L_n(A_n')+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "1+H_{n-1}(a_n')+1"] \\
M_{n+1}H_n(A_n)+G_nL_{n+1}(X_{n+1})+G_n(A_n') \arrow[d, "1+\mu_{n+1X_{n+1}}+1"'] & M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+H_{n-1}(0)+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\simeq"] \\
M_{n+1}H_n(A_n)+M_{n+1}G_{n+1}(X_{n+1})+G_n(A_n') \arrow[d, "\simeq"'] \arrow[r, phantom, "\mathrm{V}"] & M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "M_{n+1}H_n(x_{n+1}^{-1})+1"] \arrow[uuul, "M_{n+1}H_n(x_{n+1}^{-1})+G_n(x_{n+1}^{-1})"'] \\
M_{n+1}(H_n(A_n)+G_{n+1}(X_{n+1}))+G_n(A_n') & M_{n+1}H_n(A_n)+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\simeq"] \\
 & M_{n+1}H_n(A_n)+G_nL_{n+1}(X_{n+1})+G_n(A_n') \arrow[d, "1+\mu_{n+1X_{n+1}}+1"] \\
 & M_{n+1}H_n(A_n)+M_{n+1}G_{n+1}(X_{n+1})+G_n(A_n') \arrow[uul, "\simeq"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+H_{n-1}L_n(A_n')+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[dr, "\simeq"] \arrow[dd, "1+H_{n-1}(a_n')+1"'] & \\
 & M_{n+1}H_nL_{n+1}(X_{n+1})+G_nL_{n+1}(X_{n+1})+(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n') \arrow[d, "1+\mu_{n+1X_{n+1}}+1"] \\
M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+H_{n-1}(0)+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[dd, "\simeq"'] & M_{n+1}H_nL_{n+1}(X_{n+1})+M_{n+1}G_{n+1}(X_{n+1})+(M_{n+1}H_n+H_{n-1}L_n+G_n)(A_n') \arrow[d, "\simeq"] \\
 & M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+M_{n+1}G_{n+1}(X_{n+1})+H_{n-1}L_n(A_n')+G_n(A_n') \arrow[dd, "M_{n+1}H_n(x_{n+1}^{-1})+1+H_{n-1}(a_n')+1"] \\
M_{n+1}H_n(L_{n+1}(X_{n+1})+A_n')+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "M_{n+1}H_n(x_{n+1}^{-1})+1"'] & \\
M_{n+1}H_n(A_n)+G_n(L_{n+1}(X_{n+1})+A_n') \arrow[d, "\simeq"'] \arrow[r, phantom, "\mathrm{VI}"] & M_{n+1}H_n(A_n)+M_{n+1}G_{n+1}(X_{n+1})+H_{n-1}(0)+G_n(A_n') \arrow[d, "\simeq"] \\
M_{n+1}H_n(A_n)+G_nL_{n+1}(X_{n+1})+G_n(A_n') \arrow[d, "1+\mu_{n+1X_{n+1}}+1"'] & M_{n+1}H_n(A_n)+M_{n+1}G_{n+1}(X_{n+1})+G_n(A_n') \arrow[d, "\simeq"] \\
M_{n+1}H_n(A_n)+M_{n+1}G_{n+1}(X_{n+1})+G_n(A_n') \arrow[r, "\simeq"] & M_{n+1}(H_n(A_n)+G_{n+1}(X_{n+1}))+G_n(A_n')
\end{tikzcd}
$$

$$
\begin{tikzcd}
F_nL_{n+1}(X_{n+1}) \arrow[r, "\lambda_{n+1X_{n+1}}"] \arrow[d, "\tau_{nL_{n+1}(X_{n+1})}"'] & M_{n+1}F_{n+1}(X_{n+1}) \arrow[d, "M_{n+1}(\tau_{n+1X_{n+1}})"] \\
(M_{n+1}H_n + H_{n-1}L_n + G_n)L_{n+1}(X_{n+1}) \arrow[d, "\simeq"'] \arrow[r, phantom, "\mathrm{VII}"] & M_{n+1}(M_{n+2}H_{n+1} + H_nL_{n+1} + G_{n+1})(X_{n+1}) \arrow[d, "\simeq"] \\
M_{n+1}H_nL_{n+1}(X_{n+1}) + H_{n-1}L_nL_{n+1}(X_{n+1}) + G_nL_{n+1}(X_{n+1}) \arrow[d, "1+H_{n-1}(\alpha_{n+1X_{n+1}})+1"'] & M_{n+1}M_{n+2}H_{n+1}(X_{n+1}) + M_{n+1}H_nL_{n+1}(X_{n+1}) + M_{n+1}G_{n+1}(X_{n+1}) \arrow[d, "\beta_{n+2H_{n+1}(X_{n+1})}+1+1"] \\
M_{n+1}H_nL_{n+1}(X_{n+1}) + H_{n-1}(0) + G_nL_{n+1}(X_{n+1}) \arrow[d, "\simeq"'] & 0 + M_{n+1}H_nL_{n+1}(X_{n+1}) + M_{n+1}G_{n+1}(X_{n+1}) \arrow[d, "\simeq"] \\
M_{n+1}H_nL_{n+1}(X_{n+1}) + G_nL_{n+1}(X_{n+1}) \arrow[r, "1+\mu_{n+1X_{n+1}}"] & M_{n+1}H_nL_{n+1}(X_{n+1}) + M_{n+1}G_{n+1}(X_{n+1})
\end{tikzcd}
$$

where I is commutative because τ_n is a natural transformation. II, III follow from the properties of symmetric monoidal functors. IV follows from operation of H_{n-1} on the commutative diagram of $[X_{n+1}, x_{n+1}]$. V and VI follow from the properties of symmetric 2-groups. VII follows from the commtutative diagram of τ_{n+1}.

Moreover, for any two objects (A_n, a_n), $(A_n^{'}, a_n^{'})$ of $\mathcal{H}_n(\mathcal{A}.)$, $(A_n, a_n)+(A_n^{'}, a_n^{'}) = (A_n + A_n^{'}, a_n + a_n^{'})$ with $L_{n-1}(a_n + a_n^{'}) = (\alpha_n)_{A_n+A_n'}$, we have the following commutative diagram

$$
\begin{tikzcd}
\mathcal{H}_n(F.)((A_n,a_n)+(A_n^{'},a_n^{'})) \arrow[r, "\simeq"] \arrow[d, "\varphi_{n_{(A_n,a_n)+(A_n^{'},a_n^{'})}} = {[H_n(A_n+A_n^{'}), \overline{y_{n+1}}]}"'] & \mathcal{H}_n(F.)(A_n,a_n)+\mathcal{H}_n(F.)(A_n^{'},a_n^{'}) \arrow[d, "\varphi_{n(A_n,a_n)} + \varphi_{n(A_n^{'},a_n^{'})} = {[H_n(A_n), y_{n+1}]} + {[H_n(A_n^{'}), y_{n+1}^{'}]}"] \\
\mathcal{H}_n(G.)((A_n,a_n)+(A_n^{'},a_n^{'})) \arrow[r, "\simeq"'] & \mathcal{H}_n(G.)(A_n,a_n)+\mathcal{H}_n(G.)(A_n^{'},a_n^{'})
\end{tikzcd}
$$

where $\overline{y_{n+1}}, y_{n+1}, y_{n+1}^{'}$ are induced by τ_n as above. In fact, $[H_n(A_n), y_{n+1}] + [H_n(A_n^{'}), y_{n+1}^{'}] = [H_n(A_n)+H_n(A_n^{'}), y_{n+1}+y_{n+1}^{'}]$, so $(\varphi_n)_{(A_n,a_n)+(A_n',a_n')} = (\varphi_n)_{(A_n,a_n)} + (\varphi_n)_{(A_n',a_n')}$, then the above diagram commutes.

Then from above, we proved φ_n is a 2-morphism in (2-SGp), for each n. $\square$

From the definition of 2-functors, we have the following Lemma.

Lemma 1. *Let T :(2-SGp)$\to$(2-SGp) be a 2-functor, $(F., \lambda.), (G., \mu.) : (\mathcal{A}., L., \alpha.) \to (\mathcal{B}., M., \beta.)$ be two 2-chain homotopy morphisms of complexes in (2-SGp). Then $T(F., \lambda.)$ is 2-chain homotopic to $T(G., \mu.)$ in (2-SGp).*

3 Projective resolution of symmetric 2-groups

In this section we will give the construction of projective resolution of any symmetric 2-group.

Definition 3. Let $\mathcal{M}$ be a symmetric 2-group. A projective resolution of $\mathcal{M}$ in (2-SGp) is a 2-chain complex of symmetric 2-groups which is relative 2-exact in each point as in the following diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "F_3"] & \mathcal{P}_2 \arrow[r, "F_2"] \arrow[rr, bend left=40, "0"] & \mathcal{P}_1 \arrow[r, "F_1"] \arrow[rr, bend right=40, "0"'] \arrow[u, phantom, "\Uparrow\alpha_2"] & \mathcal{P}_0 \arrow[r, "F_0"] \arrow[rr, bend left=40, "0"] \arrow[d, phantom, "\Downarrow\alpha_1"] & \mathcal{M} \arrow[r, "0"] \arrow[u, phantom, "\Uparrow can"] & 0
\end{tikzcd}
$$

with $\mathcal{P}_n (n \geq 0)$ projective objects in (2-SGp). i.e. the above complex is relative 2-exact in each $\mathcal{P}_i$ and $\mathcal{M}$.

Proposition 2. *Every symmetric 2-group $\mathcal{M}$ has a projective resolution in (2-SGp).*

Proof. We will construct the projective resolution of $\mathcal{M}$ using the relative kernel.

For $\mathcal{M}$, there is an essentially surjective morphism $F_0 : \mathcal{P}_0 \to \mathcal{M}$, with $\mathcal{P}_0$ projective object in (2-SGp)([13, 25]). Then we get a sequence as follows

$$
\begin{tikzcd}
\mathcal{P}_0 \arrow[r, "F_0"] \arrow[rr, bend left=40, "0"] & \mathcal{M} \arrow[r, "0"] \arrow[u, phantom, "\Uparrow can"] & 0
\end{tikzcd}
\quad \text{S.1.}
$$

where $0 : \mathcal{M} \to 0$ is the zero morphism[9, 24]in (2-SGp), 0 is the symmetric 2-group with only one object and one morphism., *can* is the canonical 2-morphism in (2-SGp), which is given by the identity morphism of only one object of 0.

From the existence of the relative kernel in (2-SGp), we have the relative kernel $(Ker(F_0, can),\ e_{(F_0,can)},\ \varepsilon_{(F_0,can)})$ of the sequence S.1, which is in fact the general kernel $(KerF_0, e_{F_0}, \varepsilon_{F_0})$ [1]. For the symmetric 2-group $KerF_0$, there exists an essentially surjective morphism $G_1 : \mathcal{P}_1 \to KerF_0$, with $\mathcal{P}_1$ projective object in

(2-SGp)([13, 25]). Let $F_1 = e_{F_0} \circ G_1 : \mathcal{P}_1 \to \mathcal{P}_0$. Then we get the following sequence

$$\begin{tikzcd}
\mathcal{P}_1 \arrow[r, "F_1"] \arrow[rr, bend left=50] \arrow[rr, phantom, bend left=25, "\alpha_1\Uparrow"] \arrow[dr, "G_1"'] & \mathcal{P}_0 \arrow[r, "F_0"] \arrow[dl, phantom, near start, "\Uparrow id"] \arrow[dr, phantom, near start, "\Downarrow \varepsilon_{F_0}"] & \mathcal{M} \arrow[r, "0"] & 0 \\
& KerF_0 \arrow[u, "e_{F_0}"] \arrow[ur, "0"'] & &
\end{tikzcd}$$

where α_1 is the composition $F_0 \circ F_1 = F_0 \circ e_{F_0} \circ G_1 \Rightarrow 0 \circ G_1 \Rightarrow 0$ and compatible with can.

Consider the above sequence, there exists the relative kernel $(Ker(F_1, \alpha_1), e_{(F_1,\alpha_1)}, \varepsilon_{(F_1,\alpha_1)})$ in (2-SGp). For the symmetric 2-group $Ker(F_1, \alpha_1)$, there is an essentially surjective morphism $G_2 : \mathcal{P}_2 \to Ker(F_1, \alpha_1)$, with $\mathcal{P}_2$ projective object in (2-SGp)([13, 25]). Let $F_2 = e_{(F_1,\alpha_1)} \circ G_2 : \mathcal{P}_2 \to \mathcal{P}_1$. Then we get a sequence

$$\begin{tikzcd}
& Ker(F_1,\alpha_1) \arrow[d, "e_{(F_1\alpha_1)}"'] \arrow[dr, "0"] \arrow[dr, phantom, near start, "\varepsilon_{(F_1\alpha_1)} \Rightarrow"'] & & & \\
\mathcal{P}_2 \arrow[ur, "G_2"] \arrow[r, "F_2"] \arrow[rr, bend right=50, "0"'] & \mathcal{P}_1 \arrow[r, "F_1"] \arrow[rr, bend right=50, "0"'] \arrow[d, phantom, "\alpha_2\Downarrow"] & \mathcal{P}_0 \arrow[r, "F_0"] \arrow[d, phantom, "\Downarrow\alpha_1"] & \mathcal{M} \arrow[r, "0"] & 0 \\
& {} & {} & &
\end{tikzcd}$$

where α_2 is the composition $F_1 \circ F_2 = F_1 \circ e_{(F_1,\alpha_1)} \circ G_2 \Rightarrow 0 \circ G_2 \Rightarrow 0$ and compatible with α_1.

Using the same method, we get a 2-chian complex of symmetric 2-groups

$$\begin{tikzcd}
\cdots \arrow[r, "F_3"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\alpha_2\Uparrow"] & \mathcal{P}_2 \arrow[r, "F_2"] & \mathcal{P}_1 \arrow[r, "F_1"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\alpha_1\Downarrow"] & \mathcal{P}_0 \arrow[r, "F_0"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow can"] & \mathcal{M} \arrow[r, "0"] & 0
\end{tikzcd}$$

Fig.2.

Next, we will check that the complex Fig.2 is relative 2-exact in each point.

Firstly, the complex Fig.2 is relative 2-exact in $\mathcal{M}$. In fact, F_0 is essentially surjective([1]).

Secondly, the complex Fig.2 is relative 2-exact in $\mathcal{P}_0$. From the cancellation property of e_{F_0}, there exists $\overline{\alpha_2} : G_1 \circ F_2 \Rightarrow 0$ defined by $\overline{(\alpha_2)_y} \triangleq (\alpha_2)_y : G_1F_2(y) \to 0$, $\forall y \in obj(\mathcal{P}_2)$. And $G_1 : \mathcal{P}_1 \to KerF_0$ is in fact $G_1(x) = (F_1(x), (\alpha_1)_x)$. For any $x_1, x_2 \in obj(\mathcal{P}_1)$ and the morphism $g : G_1(x_1) \to G_1(x_2)$ of $KerF_0$. Under the morphism $e_{F_0} : KerF_0 \to \mathcal{P}_0$, we have a morphism $e_{F_0}(g) : e_{F_0}G_1(x_1) = F_1(x_1) \to e_{F_0}G_1(x_2) = F_1(x_2)$ of $\mathcal{P}_0$, then we get a composition morphism $e_{F_0}(g) + 1 : F_1(x_1 + x_2^*) \backsimeq F_1(x_1) + F_1(x_2)^* \to F_1(x_2) + F_1(x_2)^* \backsimeq 0$ and a commutative diagram

$$\begin{tikzcd}
F_0F_1(x_1+x_2^*) \arrow[rr, "F_0(e_{F_0}(g+1))"] \arrow[dr, "\alpha_{1_{x_1+x_2^*}}"'] & & F_0(0) \arrow[dl] \\
& 0 &
\end{tikzcd}$$

by the compatibility of ε_{F_0} and α_1.

We get an object $(x_1 + x_2^*, e_{F_0}(g) + 1)$ in $Ker(F_1, \alpha_1)$, and from the essentially surjective morphism $G_2 : \mathcal{P}_2 \to Ker(F_1, \alpha_1)$, there exist an object y in $\mathcal{P}_2$ and the isomorphism $h : (x_1 + x_2^*, e_{F_0}(g) + 1)) \to G_2(y)$ in $Ker(F_1, \alpha_1)$. Using the morphism $e_{(F_1,\alpha_1)} : Ker(F_1, \alpha_1) \to \mathcal{P}_1$, we have a morphism $e_{(F_1,\alpha_1)}(h) : x_1 + x_2^* \to e_{(F_1,\alpha_1)}G_2(y) = F_2(y)$. So we have a morphism

$$f : x_1 \to x + 0 \to x_1 + (x_2^* + x_2) \to (x_1 + x_2^*) + x_2 \to F_2(y) + x_2,$$

such that

$$\begin{tikzcd}
G_1(x_1) \arrow[rr, "G_1(f)"] \arrow[dr, "g"'] & & G_1(F_2(y)+x_2) \arrow[dl, "\bar{\alpha}_{2_y}+1"] \\
& G_1(x_2) &
\end{tikzcd}$$

So we proved that the essentially surjective morphism G_1 is $\overline{\alpha_2}$-full, the complex Fig.2 is relative 2-exact in $\mathcal{P}_0$.

Using the same method, we can prove the complex Fig.2 is relative 2-exact in each point. $\square$

Theorem 1. *Let $(F. : \mathcal{P}. \to \mathcal{M}, \alpha.)$ be a projective resolution of symmetric 2-group $\mathcal{M}$, and $H : \mathcal{M} \to \mathcal{N}$ a morphism in (2-SGp). Then for any projective resolution*

$(G_. : \mathcal{Q}_. \to \mathcal{N}, \beta_.)$, there is a morphism $H_. : \mathcal{P}_. \to \mathcal{Q}_.$ of complexes in (2-SGp) together with the family of 2-morphisms $\{\varepsilon_n : G_n \circ H_n \Rightarrow H_{n-1} \circ F_n\}_{n \geq 0}$(where $H_{-1} = H$) as in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r, "F_3"] & \mathcal{P}_2 \arrow[r, "F_2"] \arrow[d] \arrow[rr, bend left, "0"] \arrow[dr, phantom, "\varepsilon_2"] & \mathcal{P}_1 \arrow[r, "F_1"] \arrow[d] \arrow[rr, bend left, "0"] \arrow[u, phantom, "\alpha_2 \Uparrow"] \arrow[dr, phantom, "\varepsilon_1"] & \mathcal{P}_0 \arrow[r, "F_0"] \arrow[d] \arrow[rr, bend left, "0"] \arrow[u, phantom, "\alpha_1 \Uparrow"] \arrow[dr, phantom, "\varepsilon_0"] & \mathcal{M} \arrow[r, "0"] \arrow[d] \arrow[u, phantom, "\Uparrow can"] & 0 \\
\cdots \arrow[r, "G_3"] & \mathcal{Q}_2 \arrow[r, "G_2"] \arrow[rr, bend right, "0"'] & \mathcal{Q}_1 \arrow[r, "G_1"] \arrow[rr, bend right, "0"'] \arrow[d, phantom, "\beta_2 \Downarrow"] & \mathcal{Q}_0 \arrow[r, "G_0"] \arrow[rr, bend right, "0"'] \arrow[d, phantom, "\beta_1 \Downarrow"] & \mathcal{N} \arrow[r, "0"] \arrow[d, phantom, "\Downarrow can"] & 0
\end{tikzcd}$$

If there is another morphism between projective resolutions, they are 2-chain homotopy.

Proof. The existence of $H_0 : \mathcal{P}_0 \to \mathcal{Q}_0$: Since G_0 is essentially surjective and $\mathcal{P}_0$ is a projective object in (2-SGp), there exist 1-morphism $H_0 : \mathcal{P}_0 \to \mathcal{Q}_0$ and 2-morphism $\varepsilon_0 : G_0 \circ H_0 \Rightarrow H \circ F_0$ as follows

$$\begin{tikzcd}
 & \mathcal{P}_0 \arrow[dl, dashed, "H_0"'] \arrow[d, "H \circ F_0"] \\
\mathcal{Q}_0 \arrow[r, "G_0"'] \arrow[ur, phantom, "\varepsilon_0"] & \mathcal{N}
\end{tikzcd}$$

Consider the morphism $H_0 : \mathcal{P}_0 \to \mathcal{Q}_0$, we have a morphism

$$\begin{aligned}
\overline{H_0} : KerF_0 &\to KerG_0 \\
(x_0, a_0) &\mapsto (H_0(x_0), \widetilde{a_0}), \\
(x_0, a_0) \xrightarrow{f_0} (x_0', a_0') &\mapsto (H_0(x_0), \widetilde{a_0}) \xrightarrow{H_0(f_0)} (H_0(x_0'), \widetilde{a_0}')
\end{aligned}$$

where $\widetilde{a_0}$ is the composition $(G_0 \circ H_0)(x_0) \xrightarrow{(\varepsilon_0)_{x_0}} (H \circ F_0)(x_0) \xrightarrow{H(a_0)} H(0) \backsimeq 0$. Moreover, there is a commutative diagram

$$\begin{tikzcd}
KerF_0 \arrow[r, "e_{F_0}"] \arrow[d, "\overline{H_0}"'] \arrow[dr, bend right] & \mathcal{P}_0 \arrow[d, "H_0"] \\
KerG_0 \arrow[r, "e_{G_0}"'] & \mathcal{Q}_0
\end{tikzcd}$$

From the relative 2-exactness of projective resolution of symmetric 2-group, there exist essentially surjective morphisms $\overline{F_1} : \mathcal{P}_1 \to kerF_0$, $\overline{G_1} : \mathcal{Q}_1 \to kerG_0$ and 2-morphisms $\varphi_1 : e_{F_0} \circ \overline{F_1} \Rightarrow F_1$, $\psi_1 : e_{G_0} \circ \overline{G_1} \Rightarrow G_1$, respectively. Then there exist 1-morphism $H_1 : \mathcal{P}_1 \to \mathcal{Q}_1$ and 2-morphism $\overline{\varepsilon_1} : \overline{G_1} \circ H_1 \Rightarrow \overline{H_0} \circ \overline{F_1}$. From $\overline{\varepsilon_1}$ and $e_{G_0} \circ \overline{H_0} = H_0 \circ e_{F_0}$, we can define a 2-morphism $\varepsilon_1 : G_1 \circ H_1 \Rightarrow H_0 \circ F_1$ by $(\varepsilon_1)_{x_1} : (G_1 \circ H_1)(x_1) \xrightarrow{(\psi_1)^{-1}_{H_1(x_1)}} e_{G_0} \circ \overline{G_1} \circ H_1(x_1) \xrightarrow{e_{G_0}(\overline{(\varepsilon_1)}_{x_1})} e_{G_0} \circ \overline{H_0} \circ \overline{F_1}(x_1) = H_0 \circ e_{F_0} \circ \overline{F_1}(x_1) \xrightarrow{H_0((\varphi_1)_{x_1})} H_0 \circ F_1(x_1)$, which is compatible with ε_0.

Next we will construct H_n and $\varepsilon_n : G_n \circ H_n \Rightarrow H_{n-1} \circ F_n$ by induction on n. Inductively, suppose H_i and ε_i have been constructed for $i \leq n$ satisfying the compatible conditions. Consider the morphism $H_{n-1} : \mathcal{P}_{n-1} \to \mathcal{Q}_{n-1}$, there is an induced morphism

$$\begin{aligned}
\overline{H_{n-1}} : Ker(F_{n-1}, \alpha_{n-1}) &\to Ker(G_{n-1}, \beta_{n-1}) \\
(x_{n-1}, a_{n-1}) &\mapsto (H_{n-1}(x_{n-1}), \widetilde{a_{n-1}}), \\
f_{n-1} &\mapsto H_{n-1}(f_{n-1})
\end{aligned}$$

where $\widetilde{a_{n-1}}$ is the composition $G_{n-1} \circ H_{n-1}(x_{n-1}) \to H_{n-2} \circ F_{n-1}(x_{n-1}) \xrightarrow{H_{n-2}(a_{n-1})} H_{n-1}(0) \simeq 0$. Moreover, there is the following commutative diagram

$$\begin{tikzcd}
Ker(F_{n-1}, \alpha_{n-1}) \arrow[r, "e_{(F_{n-1},\alpha_{n-1})}"] \arrow[d, "\overline{H_{n-1}}"'] & \mathcal{P}_{n-1} \arrow[d, "H_{n-1}"] \\
Ker(G_{n-1}, \beta_{n-1}) \arrow[r, "e_{(G_{n-1},\beta_{n-1})}"'] & \mathcal{Q}_{n-1}
\end{tikzcd}$$

Using the relative 2-exactness of projective resolutions of $\mathcal{M}$ and $\mathcal{N}$, we have the following diagram

$$
\begin{tikzcd}
\mathcal{P}_n \arrow[r, "\overline{F_n}"] \arrow[d, "H_n"'] \arrow[rr, bend left, "F_n"] \arrow[dr, phantom, "\overline{\varepsilon_n}"] & Ker(F_{n-1},\alpha_{n-1}) \arrow[r, "e_{(F_{n-1},\alpha_{n-1})}"] \arrow[d, "\overline{H_{n-1}}"] \arrow[u, phantom, "\varphi_n\Uparrow"] & \mathcal{P}_{n-1} \arrow[d, "H_{n-1}"] \\
\mathcal{Q}_n \arrow[r, "\overline{G_n}"'] \arrow[rr, bend right, "G_n"'] & Ker(G_{n-1},\beta_{n-1}) \arrow[r, "e_{(G_{n-1},\beta_{n-1})}"'] \arrow[d, phantom, "\psi_n\Downarrow"] & \mathcal{Q}_{n-1}
\end{tikzcd}
$$

The existence of H_n and $\overline{\varepsilon_n}$ come from the projectivity of $\mathcal{P}_n$. Similar to the appearing of ε_1, there is a 2-morphism ε_n given by $\overline{\varepsilon_n}$, compatible with ε_{n-1}.

Next, we show the uniqueness of $(H.,\varepsilon.)$ up to 2-chain homotopy. Suppose $(K.,\zeta.)$ is another morphism of projective resolutions. We will construct the 1-morphism $T_n : \mathcal{P}_n \to \mathcal{Q}_{n+1}$, and 2-morphism $\tau_n : H_n \Rightarrow G_{n+1}\circ T_n + T_{n-1}\circ F_n + K_n$ by induction on n. If $n<0$, $\mathcal{P}_n = 0$, so we get $T_n = 0$. If $n=0$, there is a 1-morphism $H_0 - K_0 : \mathcal{P}_0 \to KerG_0$, together with essentially surjective morphism $\overline{G_1} : \mathcal{Q}_1 \to KerG_0$, there exist a morphism $T_0 : \mathcal{P}_0 \to \mathcal{Q}_1$ and 2-morphism $\tau_0^{'} : \overline{G_1}\circ T_0 \Rightarrow H_0 - K_0$. Then we get a 2-morphism $\tau_0 : H_0 \Rightarrow G_1 \circ T_0 + K_0$.

Inductively, we suppose given family of morphisms $(H_i,\tau_i)_{i\leq n}$ so that $H_i : \mathcal{P}_i \to \mathcal{Q}_{i+1}, \tau_i : H_i \Rightarrow G_{i+1}\circ T_i + T_{i-1}\circ F_i + K_i$. Consider the 1-morphism $H_n - K_n - T_{n-1}\circ F_n : \mathcal{P}_n \to Ker(G_n,\beta_n)$ and essentially surjective morphism $\overline{G_{n+1}} : \mathcal{Q}_{n+1} \to Ker(G_n,\beta_n)$, there exist a 1-morphism $T_n : \mathcal{P}_n \to \mathcal{Q}_{n+1}$ and a 2-morphism $\tau_n^{'} : \overline{G_{n+1}}\circ T_n \Rightarrow H_n - K_n - T_{n-1}\circ F_n$. Then we get a 2-morphism $\tau : H_n \Rightarrow G_{n+1}\circ T_n + T_{n-1}\circ F_n + K_n$. $\square$

4 Derived 2-Functor

In this section, we will give the left derived 2-functor in the abelian 2-category (2-SGp), which has enough projective objects[13, 25].

Definition 4. An additive 2-functor([9]) T: (2-SGp)→(2-SGp) is called right relative 2-exact if the relative 2-exactness of

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow\mathrm{can}"] & \mathcal{A} \arrow[r, "F"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow\varphi"] & \mathcal{B} \arrow[r, "G"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow\mathrm{can}"] & \mathcal{C} \arrow[r, "0"] & 0
\end{tikzcd}$$

in $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ implies relative 2-exactness of

$$\begin{tikzcd}
T(\mathcal{A}) \arrow[r, "T(F)"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow T(\varphi)"] & T(\mathcal{B}) \arrow[r, "T(G)"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow\mathrm{can}"] & T(\mathcal{C}) \arrow[r, "0"] & 0
\end{tikzcd}$$

in $T(\mathcal{B})$ and $T(\mathcal{C})$.

The left relative 2-exact 2-functor can be defined dually.

By Remark 2 and Proposition 1, Theorem 1, there is

Corollary 1. *Let T :(2-SGp)$\to$(2-SGp) be an additive 2-functor, and $\mathcal{A}$ be any object of (2-SGp). For two projective resolutions $\mathcal{P}.$, $\mathcal{Q}.$ of $\mathcal{A}$, there is an equivalence between homology symmetric 2-groups $\mathcal{H}.(T(\mathcal{P}.))$ and $\mathcal{H}.(T(\mathcal{Q}.))$.*

Let T: (2-SGp)$\to$(2-SGp) be an additive 2-functor. There is a 2-functor

$$\begin{aligned}
\mathcal{L}_iT : \text{(2-SGp)} &\to \text{(2-SGp)} \\
\mathcal{A} &\mapsto \mathcal{L}_iT(\mathcal{A}), \\
\mathcal{A} \xrightarrow{F} \mathcal{B} &\mapsto \mathcal{L}_iT(\mathcal{A}) \xrightarrow{\mathcal{L}_iT(F)} \mathcal{L}_iT\mathcal{B}),
\end{aligned}$$

where $\mathcal{L}_iT(\mathcal{A})$ is defined by $\mathcal{H}_i(T(\mathcal{P}.))$, and $\mathcal{P}.$ is the projective resolution of $\mathcal{A}$. $\mathcal{L}_iT$ is a well-defined 2-functor from the properties of additive 2-functor and Corollary 1.

Corollary 2. *Let T: (2-SGp)$\to$(2-SGp) be a right relative 2-exact 2-functor, and $\mathcal{A}$ be a projective object in (2-SGp). Then $\mathcal{L}_iT(\mathcal{A}) = 0$ for $i \neq 0$.*

The following is a basic property of derived functors.

Theorem 2. *The left derived 2-functor $\mathcal{L}_*T$ takes the sequence of symmetric 2-groups*

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "0"] \arrow[dr, "F"'] & {} \arrow[d, phantom, "\varphi\Uparrow"] & \mathcal{C} \\ & \mathcal{B} \arrow[ur, "G"'] & \end{tikzcd}$$

which is relative 2-exact in $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ to a long sequence 2-exact([2, 24])in each point

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{L}_nT(\mathcal{A}) \arrow[r, "\mathcal{L}_nT(F)"] \arrow[rr, bend left, "0"] & \mathcal{L}_nT(\mathcal{B}) \arrow[r, "\mathcal{L}_nT(G)"] \arrow[u, phantom, "\Uparrow\mathcal{L}_nT(\varphi)"] \arrow[rr, bend right, "0"'] & \mathcal{L}_nT(\mathcal{C}) \arrow[r, "\Delta_n"] \arrow[rr, bend left, "0"] \arrow[d, phantom, "\Downarrow\Sigma_n"] & \mathcal{L}_{n-1}T(\mathcal{A}) \arrow[r, "\mathcal{L}_{n-1}T(F)"] \arrow[u, phantom, "\Uparrow\Psi_n"] & \mathcal{L}_{n-1}T(\mathcal{B}) \arrow[r] & \cdots \end{tikzcd}$$

In order to prove this theorem, we need the following Lemmas.

Lemma 2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be projective objects in (2-SGp). Then the product category $\mathcal{P} \times \mathcal{Q}$ is a projective object in (2-SGp).*

Proof. First we know that $\mathcal{P} \times \mathcal{Q}$ is a symmetric 2-group([9, 12]). So we need to prove the projectivity of it. There are canonical morphisms

$$\mathcal{P} \overset{p_1}{\longleftarrow} \mathcal{P} \times \mathcal{Q} \overset{p_2}{\longrightarrow} \mathcal{Q},\ \mathcal{P} \overset{i_1}{\longrightarrow} \mathcal{P} \times \mathcal{Q} \overset{i_2}{\longleftarrow} \mathcal{Q}.$$

For any morphism $G : \mathcal{P} \times \mathcal{Q} \to \mathcal{B}$, there are composition morphisms $G_1 : \mathcal{P} \overset{i_1}{\to} \mathcal{P} \times \mathcal{Q} \overset{G}{\to} \mathcal{B}$, $G_2 : \mathcal{Q} \overset{i_2}{\to} \mathcal{P} \times \mathcal{Q} \overset{G}{\to} \mathcal{B}$. Then for an essentially surjective functor $F : \mathcal{A} \to \mathcal{B}$, there exist 1-morphisms $G_1^{'} : \mathcal{P} \to \mathcal{A}$, $G_2^{'} : \mathcal{Q} \to \mathcal{A}$ and 2-morphisms $h_1 : F \circ G_1^{'} \Rightarrow G_1$, $h_2 : F \circ G_2^{'} \Rightarrow G_2$ since $\mathcal{P}$ and $\mathcal{Q}$ are projective objects in (2-SGp).

So there are 1-morphism $G^{'} : \mathcal{P} \times \mathcal{Q} \to \mathcal{A}$ given by $G^{'} \triangleq G_1^{'} \circ p_1 + G_2^{'} \circ p_2$ and 2-morphism $h : F \circ G^{'} \Rightarrow G : \mathcal{P} \times \mathcal{Q} \to \mathcal{B}$ given by the composition $h_{(x,y)} : (F \circ G^{'})(x,y) = F(G_1^{'}(x) + G_2^{'}(y)) \backsimeq F(G_1^{'}(x)) + F(G_2^{'}(y)) \xrightarrow{(h_1)_x + (h_2)_y} G_1(x) + G_2(y) = G(x,0) + G(0,y) \backsimeq G((x,0) + (0,y)) = G(x,y)$, for any $(x,y) \in obj(\mathcal{P} \times \mathcal{Q})$.

Then $\mathcal{P} \times \mathcal{Q}$ is a projective object in (2-SGp). $\square$

Lemma 3. *Let* $(F, \varphi, G) : \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ *be an extension of symmetric 2-groups in (2-SGp)([2, 1]),* $(\mathcal{P}., L., \alpha.)$ $(\mathcal{Q}., N., \beta.)$ *be projective resolutions of* $\mathcal{A}$ *and* $\mathcal{C}$*, respectively. Then there is a projective resolution* $(\mathcal{K}., M., \varphi.)$ *of* $\mathcal{B}$*, such that* $\mathcal{P}. \to \mathcal{K}. \to \mathcal{Q}.$ *forms an extension of complexes in (2-SGp).*

Proof. We give the construction of projective resolution $(\mathcal{K}., M., \varphi.)$ of $\mathcal{B}$ in the following steps.

Step 1. Since $\mathcal{Q}_0$ is a projective object, together with essentially surjective $G : \mathcal{B} \to \mathcal{C}$ and 1-morphism $N_0 : \mathcal{Q}_0 \to \mathcal{C}$, there exist 1-morphism $\overline{N_0} : \mathcal{Q}_0 \to \mathcal{B}$ and 2-morphism $h_0 : G \circ \overline{N_0} \Rightarrow N_0$. Then we can define a 1-morphism

$$\begin{aligned} M_0 : \mathcal{P}_0 \times \mathcal{Q}_0 &\to \mathcal{B} \\ (x_0, y_0) &\mapsto M_0(x_0, y_0) \triangleq F(L_0(x_0)) + \overline{N_0}(y_0), \\ (f_0, g_0) &\mapsto FL_0(f_0) + \overline{N_0}(g_0). \end{aligned}$$

Moreover, M_0 is essentially surjective. In fact, for any $B \in obj(\mathcal{B})$, we have $G(B) \in obj(\mathcal{C})$. Since $N_0 : \mathcal{Q}_0 \to \mathcal{C}$ is essentially surjective, there are $y_0 \in obj(\mathcal{Q}_0)$ and isomorphism $N_0(y_0) \to G(B)$, together with 2-morphism $h_0 : G \circ \overline{N_0} \Rightarrow N_0$. We get a composition isomorphism $G(\overline{N_0}(y_0)) \xrightarrow{(h_0)_{y_0}} N_0(y_0) \to G(B)$. Moreover, we get an isomorphism $c : G(B + \overline{N_0}(y_0)^*) \to 0$ in $\mathcal{C}$. Then we obtain an object $(B + \overline{N_0}(y_0)^*, c)$ of $KerG$. Since $(F, \varphi, G) : \mathcal{A} \to \mathcal{B} \to \mathcal{C}$ is an extension, by the definition of extension, there is an equivalence $F_0 : \mathcal{A} \to KerG$, which is essentially surjective, so there are $A \in obj(\mathcal{A})$ and isomorphism $F_0(A) \to (B + \overline{N_0}(y_0)^*, c)$. For $A \in obj(\mathcal{A})$ and essentially surjective morphism $L_0 : \mathcal{P}_0 \to \mathcal{A}$, there are $x_0 \in obj(\mathcal{P}_0)$ and isomorphism $L_0(x_0) \to A$. Then we get a composition isomorphism

$$F(L_0(x_0)) \to F(A) \to e_G(F_0(A)) \to e_G((B + \overline{N_0}(y_0)^*, c)) = B + \overline{N_0}(y_0)^*.$$

There is an isomorphism

$$F(L_0(x_0)) + \overline{N_0}(y_0) \to B.$$

Then, for any $B \in obj(\mathcal{B})$, there are $(x_0, y_0) \in obj(\mathcal{P}_0 \times \mathcal{Q}_0)$ and isomorphism $M_0(x_0, y_0) = F(L_0(x_0)) + \overline{N_0}(y_0) \to B$.

Also,

$$
\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[rr, bend left=40, "0"] & \mathcal{B} \arrow[r, "G"] \arrow[u, phantom, "\varphi\Uparrow"] & \mathcal{C} \\
\mathcal{P}_0 \arrow[u, "L_0"] \arrow[r, "i_0"'] \arrow[rr, bend right=40, "0"'] \arrow[ur, phantom, "\Downarrow\lambda_0"] & \mathcal{P}_0 \times \mathcal{Q}_0 \arrow[u, "M_0"'] \arrow[r, "p_0"'] \arrow[ur, phantom, "\Downarrow\mu_0"] \arrow[d, phantom, "id\,\|"] & \mathcal{Q}_0 \arrow[u, "N_0"'] \\
 & {} &
\end{tikzcd}
$$

is the morphism of extensions in (2-SGp), where $\lambda_0 : F \circ L_0 \Rightarrow M_0 \circ i_0$ is given by $(\lambda_0)_{x_0} : F(L_0(x_0) \backsimeq F(L_0(x_0)) + 0 \backsimeq F(L_0(x_0)) + \overline{N_0}(0) = M_0(x_0, 0) = M_0(i_0(x_0))$, for all $x_0 \in obj(\mathcal{P}_0)$. $\mu_0 : G \circ M_0 \Rightarrow N_0 \circ p_0$ is given by $(\mu_0)_{(x_0,y_0)} : (G \circ M_0)(x_0, y_0) = G(FL_0(x_0) + \overline{N_0}(y_0)) \backsimeq G(FL_0(x_0)) + G(\overline{N_0}(y_0)) \xrightarrow{\varphi_{L_0(x_0)} + (h_0)_{y_0}} N_0(y_0)$.

Step 2. From the definition of relative 2-exactness, there are essentially surjective 1-morphisms $L_1' : \mathcal{P}_1 \to KerL_0$, $N_1' : \mathcal{Q}_1 \to KerN_0$ as in the following diagram

$$
\begin{tikzcd}
\mathcal{P}_1 \arrow[r, "L_1'"] \arrow[d, "i_1"'] & KerL_0 \arrow[r, "e_{L_0}"] \arrow[d, "\overline{i_0}"'] & \mathcal{P}_0 \arrow[d, "i_0"] \\
\mathcal{P}_1 \times \mathcal{Q}_1 \arrow[r, "M_1'"] \arrow[d, "p_1"'] & KerM_0 \arrow[r, "e_{M_0}"] \arrow[d, "\overline{p_0}"'] & \mathcal{P}_0 \times \mathcal{Q}_0 \arrow[d, "p_0"] \\
\mathcal{Q}_1 \arrow[r, "N_1'"'] \arrow[ur, dashed, "\overline{N_1}"] \arrow[ur, phantom, "h_1\Downarrow"', near end] & KerN_0 \arrow[r, "e_{N_0}"'] & \mathcal{Q}_0
\end{tikzcd}
$$

where $M_1' : \mathcal{P}_1 \times \mathcal{Q}_1 \to KerM_0$ is given by $M_1'(x_1, y_1) \triangleq (\overline{i_0} \circ L_1')(x_1) + \overline{N_1}(y_1)$, for any $(x_1, y_1) \in obj(\mathcal{P}_1 \times \mathcal{Q}_1)$, which is essentially surjective from the proof of step 1.

Then we get a composition 1-morphism $M_1 = e_{M_0} \circ M_1' : \mathcal{P}_1 \times \mathcal{Q}_1 \to \mathcal{P}_0 \times \mathcal{Q}_0$, and a composition 2-morphism $\varphi_1 : M_0 \circ M_1 \Rightarrow 0 \circ M_1' \Rightarrow 0$, such that

$$
\begin{tikzcd}
 & {} & \\
\mathcal{P}_0 \arrow[r, "i_0"] \arrow[rr, bend left=40, "0"] & \mathcal{P}_0 \times \mathcal{Q}_0 \arrow[r, "p_0"] \arrow[u, phantom, "\|\,id"] & \mathcal{Q}_0 \\
\mathcal{P}_1 \arrow[u, "L_1"] \arrow[r, "i_1"'] \arrow[rr, bend right=40, "0"'] \arrow[ur, phantom, "\Rrightarrow\lambda_1"] & \mathcal{P}_1 \times \mathcal{Q}_1 \arrow[u, "M_1"'] \arrow[r, "p_1"'] \arrow[ur, phantom, "\Rrightarrow\mu_1"] \arrow[d, phantom, "\|\,id"] & \mathcal{Q}_1 \arrow[u, "N_1"'] \\
 & {} &
\end{tikzcd}
$$

is a morphism of extensions in (2-SGp), where λ_1 and μ_1 are given in the natural way as in step 1.

Step 3. From the definition of relative 2-exactness, there are essentially surjective 1-morphisms $L_2^{'} : \mathcal{P}_2 \to Ker(L_1, \alpha_1)$, $N_2^{'} : \mathcal{Q}_2 \to Ker(N_1, \beta_1)$ as in the following diagram

$$
\begin{tikzcd}
\mathcal{P}_2 \arrow[r, "L_2^{'}"] \arrow[d, "i_2"'] & Ker(L_1,\alpha_1) \arrow[r, "e_{(L_1,\alpha_1)}"] \arrow[d, "\overline{i_1}"] & \mathcal{P}_1 \arrow[d, "i_1"] \\
\mathcal{P}_1 \times \mathcal{Q}_1 \arrow[r, "M_2^{'}"] \arrow[d, "p_2"'] & Ker(M_1,\varphi_1) \arrow[r, "e_{(M_1,\varphi_1)}"] \arrow[d, "\overline{p_1}"] & \mathcal{P}_0 \times \mathcal{Q}_0 \arrow[d, "p_1"] \\
\mathcal{Q}_1 \arrow[r, "N_2^{'}"'] \arrow[ur, dashed, "\overline{N_2}"] \arrow[ur, phantom, "h_2 \Swarrow"'] & Ker(N_1,\beta_1) \arrow[r, "e_{(N_1,\beta_1)}"'] & \mathcal{Q}_0
\end{tikzcd}
$$

where $M_2^{'} : \mathcal{P}_2 \times \mathcal{Q}_2 \to Ker(M_1, \varphi_1)$ is given by $M_2^{'}(x_2, y_2) \triangleq (\overline{i_1} \circ L_2^{'})(x_2) + \overline{N_2}(y_2)$, for any $(x_2, y_2) \in obj(\mathcal{P}_2 \times \mathcal{Q}_2)$, which is essentially surjective from the proof of step 1.

Then we get a composition 1-morphism $M_2 = e_{(M_1,\varphi_1)} \circ M_2^{'} : \mathcal{P}_2 \times \mathcal{Q}_2 \to \mathcal{P}_1 \times \mathcal{Q}_1$, and a composition 2-morphism $\varphi_2 : M_1 \circ M_2 \Rightarrow 0 \circ M_2^{'} \Rightarrow 0$, such that

$$
\begin{tikzcd}
\mathcal{P}_1 \arrow[r, "i_1"] \arrow[rr, bend left, "\overset{0}{\|\, id}"] & \mathcal{P}_1 \times \mathcal{Q}_1 \arrow[r, "p_1"] & \mathcal{Q}_1 \\
\mathcal{P}_2 \arrow[r, "i_2"'] \arrow[u, "L_2"] \arrow[rr, bend right, "\underset{0}{\|\, id}"'] \arrow[ur, phantom, "\lambda_2"] & \mathcal{P}_2 \times \mathcal{Q}_2 \arrow[r, "p_2"'] \arrow[u, "M_2"] \arrow[ur, phantom, "\mu_2"] & \mathcal{Q}_2 \arrow[u, "N_2"']
\end{tikzcd}
$$

is a morphism of extensions in (2-SGp), where λ_2 and μ_2 are given in the natural way as in step 1.

Using the same method, we get a complex $(\mathcal{P} \times \mathcal{Q}., M., \varphi.)$ of product symmetric 2-groups. Using the methods in Proposition 2, this complex is relative 2-exact in each point, and $(i., id, p.) : \mathcal{P}. \to \mathcal{P}. \times \mathcal{Q}. \to \mathcal{Q}.$ forms an extension of complexes in (2-SGp).

Set $\mathcal{K}_n = \mathcal{P}_n \times \mathcal{Q}_n$, for $n \geq 0$, which are projective objects in (2-SGp) by Lemma 3.This finishes the proof.

$\square$

By the universal property of (bi)product of symmetric 2-groups and the property of additive 2-functor([9]). We get

Lemma 4. *Let T be an additive 2-functor in 2-category (2-SGp), and $\mathcal{A}$, $\mathcal{B}$ be objects in (2-SGp). Then there is an equivalence between $T(\mathcal{A} \times \mathcal{B})$ and $T(\mathcal{A}) \times T(\mathcal{B})$ in (2-SGp).*

Proof of Theorem 2. For symmetric 2-groups $\mathcal{A}$ and $\mathcal{C}$, choose projective resolutions $\mathcal{P}. \to \mathcal{A}$ and $\mathcal{Q}. \to \mathcal{C}$. By Lemma 2 and Lemma 3, there is a projective resolution $\mathcal{P}. \times \mathcal{Q}. \to \mathcal{B}$ fitting into an extension $\mathcal{P}. \xrightarrow{i.} \mathcal{P}. \times \mathcal{Q}. \xrightarrow{p.} \mathcal{Q}.$ of projective complexes in (2-SGp)([2]). By Lemma 4, we obtain a complexes of extension

$$T(\mathcal{P}.) \xrightarrow{T(i.)} T(\mathcal{P}. \times \mathcal{Q}.) \xrightarrow{T(p.)} T(\mathcal{Q}.).$$

Similar as Theorem 4.2 in [1], the long sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{L}_nT(\mathcal{A}) \arrow[r, "\mathcal{L}_nT(F)"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow \mathcal{L}_nT(\varphi)"] & \mathcal{L}_nT(\mathcal{B}) \arrow[r, "\mathcal{L}_nT(G)"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow \Sigma_n"'] & \mathcal{L}_nT(\mathcal{C}) \arrow[r, "\Delta_n"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow \Psi_n"] & \mathcal{L}_{n-1}T(\mathcal{A}) \arrow[r, "\mathcal{L}_{n+1}T(F)"] & \mathcal{L}_{n-1}T(\mathcal{B}) \arrow[r] & \cdots
\end{tikzcd}$$

is 2-exact in each point.

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